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Spinning gas clouds: III. Solutions of minimal energy with precession

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Abstract

We consider the model of rotating and expanding gas cloud originally proposed by Ovsiannikov (1956 *Dokl. Akad. Nauk SSSR* **111** 47) and Dyson (1968 *J. Math. Mech.* **18** 91). Under the restricting assumptions of an adiabatic index $\gamma = 5/3$ and of vorticity-free motion, this has been shown (Gaffet 2001 *J. Phys. A: Math. Gen.* **34** 2097) to be a Liouville integrable Hamiltonian system. In the present work, we consider the precessing solutions where the cloud does not retain a fixed rotation axis. Choosing for definiteness a particular set of constants of motion (which corresponds to a minimum of the energy), we show that a separation of variables occurs, and that the equations of motion are reducible to the form of a Riccati equation, whose integration merely involves an elliptic integral.

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1. Introduction

We consider the class of rotating ellipsoidal gas clouds first proposed by Ovsiannikov (1956) and Dyson (1968). Gaffet (1996, 2001a: hereafter, paper I) has shown that, when the gas is monatomic (adiabatic index $\gamma = 5/3$) and there is no vorticity, the corresponding equations of gas motion constitute a Liouville integrable Hamiltonian system with five degrees of freedom. The evolution of the cloud may be described by the motion of a particle in the five-dimensional space $S_2 \times 0(3)$, where S_2 is the unit sphere and 0(3) is the three-dimensional rotation group. The motions in S_2 and 0(3) describe the evolution of the cloud's shape and orientation, respectively (there is an additional 0(3) group associated with the vorticity, but it need not be considered in the present case, where the vorticity is taken to be zero).

There are five commuting integrals of motion: the energy constant m, the total angular momentum $\vec{J}^2$, two additional integrals denoted I_6 and L_6 and one component (J_z) of angular momentum; which makes this Hamiltonian system a Liouville integrable one.

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Detailed results concerning the form of the solutions are available under the simplifying assumption of rotation around a fixed principal axis (Gaffet 2001b: paper II).

We present here the first detailed solution of the system in cases of rotation *with precession*. For simplicity, we will restrict our consideration to the minimal energy cases (where m takes its lowest possible value, the other integrals of motion being kept fixed); and, for definiteness, we choose the following set of constants:

$$\vec{J}^2 = 12 \qquad -I_6/108 = \varepsilon = 4 \qquad L_6 = 0.$$

As mentioned in paper II the corresponding minimum value of the energy is $m_0 = 5$.

The value $L_6 = 0$ is compatible with a particular solution without precession, which has already been studied in earlier works; but besides it, there exists a whole one-parameter family of solutions with precession, whose study is the subject of the present paper.

The system is reducible to one of second order, for two unknown functions $\lambda(u)$ and $w(u)$ say. We show that, under the above restricting assumptions, the system is *separable*, in the sense that the evolution of the variable w is governed by an equation involving w only

$$w'(u) = \sqrt{P(w)}$$

where P is a polynomial of the fourth degree, i.e. w is *an elliptic function* of the independent variable u. To complete the solution, the variable λ is shown to satisfy the Riccati equation

$$\mathrm{d}\lambda/\mathrm{d}w = a(w)\lambda^2 + b(w)\lambda + c(w).$$

The special solution (which is precession-free in cases where $L_6 = 0$) turns out to provide two particular solutions of the Riccati equation, whose integration is thereby reduced to one quadrature; that quadrature is an elliptic integral, which is calculable by sigma functions.

2. The equations of motion

Let us briefly recall the form of the equations of motion for a vorticity-free monatomic ellipsoidal gas cloud without preferred axis of rotation. Denoting D_1, D_2, D_3 the principal axes of the cloud, normalized so that the product $D_1D_2D_3 = 1$, we introduced in paper I a pair of variables (X_0, Y_0):

$$\begin{cases} X_0 = \mathrm{Tr}(D^2) \\ Y_0 = \mathrm{Tr}(D^{-2}) \end{cases} \tag{2.1}$$

where D is the diagonal matrix diag (D_1, D_2, D_3): (X_0, Y_0) may be viewed as a coordinate system on the unit sphere (S_2). The independent variable u that will be used here throughout (and with respect to which the Painlevé property has been shown to hold, at least in precessionless cases) is related to the Hamiltonian time t by

$$\mathrm{d}u = X_0\,\mathrm{d}t. \tag{2.2}$$

The derivative of D defines the diagonal part of a 3×3 symmetric matrix v:

$$v_{ii} = \frac{\mathrm{d}}{\mathrm{d}u} \ln D_i \qquad (i = 1, 2, 3) \tag{2.3}$$

whose off-diagonal part is related to the components of the angular momentum $\vec{j}$ in the rotating frame:

$$v_{ij} = j_k \big/ \left(D_i^2 - D_j^2\right) \tag{2.4}$$

(where (i, j, k) is a circular permutation of (1, 2, 3)).

We then reformulated the system in terms of eight physical variables: (a) $(X_{0,1,2}; Y_{0,1,2})$, is defined by

$$\begin{cases} X_n = \mathrm{Tr}(D v^n D) \\ Y_n = \mathrm{Tr}(D^{-1} v^n D^{-1}) \end{cases} \qquad (n = 0, 1, 2) \tag{2.5}$$

and (b) (T, P), the characteristic coefficients of the matrix v:

$$v^3 + Tv - P = 0. \tag{2.6}$$

The resulting equations of motion are the following:

$$\begin{cases} \text{(a)} \quad T'(u) = 3P - Y_1 \\ \text{(b)} \quad P'(u) = -\frac{2}{3}T^2 + \left(\frac{2}{3}TY_0 + Y_2\right) \end{cases} \tag{2.7}$$

$$\begin{cases} \text{(a)} \quad X_0'(u) = 2X_1 \\ \text{(b)} \quad X_1'(u) = \left(X_2 - \frac{2}{3}TX_0\right) + \left(3 - \frac{X_0 Y_0}{3}\right) \\ \text{(c)} \quad X_2'(u) = -\frac{4}{3}TX_1 - \frac{2}{3}Y_0 X_1 \end{cases} \tag{2.8}$$

$$\begin{cases} \text{(a)} \quad Y_0'(u) = -2Y_1 \\ \text{(b)} \quad Y_1'(u) = -\left(3Y_2 + \frac{2}{3}TY_0\right) + 2\left(\frac{Y_0^2}{3} - X_0\right) \\ \text{(c)} \quad Y_2'(u) = 4\left(\frac{2}{3}TY_1 - PY_0\right) + 2\left(\frac{2}{3}Y_0 Y_1 + X_1\right). \end{cases} \tag{2.9}$$

They admit the integrals of energy (m) and of total angular momentum ($\vec{j}^2$) (also denoted α^2):

$$9m = \left(X_0 X_2 - X_1^2\right) + 3X_0 \tag{2.10}$$

$$\vec{j}^2 \equiv \alpha^2 = \left(X_0 X_2 - X_1^2\right) + 3Y_2 + 4TY_0. \tag{2.11}$$

We introduced the symmetry ($\tilde{T}$) which consists of the inversion of the principal axes of the cloud

$$\tilde{D}_i = 1/D_i \qquad (i = 1, 2, 3) \tag{2.12}$$

without affecting the matrix v. While that is only a partial symmetry, not fully respected by all the equations of motion, we have noted that it is still useful to consider. It turns the angular momentum vector $\vec{j}$ into a new vector $\tilde{\vec{j}}$

$$\tilde{j}_i = -D_i^2 j_i \qquad (i = 1, 2, 3). \tag{2.13}$$

In addition to the (conserved) angular momentum $\vec{j}^2$, the quantities $A_{12} = -\vec{j} \cdot \tilde{\vec{j}}$ and $A_{22} = \tilde{\vec{j}}^2$, although not conserved, turn out to play an essential role as well, as will be seen below; they are given by expressions analogous to equation (2.11). (Note in particular the exact symmetry of the expression (2.11) of angular momentum and of A_{22}, under the exchange of X and Y; and the auto-symmetry exhibited by A_{12}.):

$$\begin{cases} A_{12} = (X_0 Y_0 + 3)T + (X_0 Y_2 + Y_0 X_2) + X_1 Y_1 \\ A_{22} = \left(Y_0 Y_2 - Y_1^2\right) + (3X_2 + 4TX_0). \end{cases} \tag{2.14}$$

The integral I_6 admits a relatively simple expression involving A_{22}

$$I_6 \equiv -108\varepsilon = 27(P + Y_1)^2 + 4T[(T + 3Y_0)^2 + 9(Y_2 - 3X_0)] + 36A_{22}. \tag{2.15}$$

(When the expression (2.14) of A_{22} is substituted in the above equation, the general expression of I_6 given in paper I (p 2104) is recovered.) Although the expression of L_6 given in paper I is complicated, it can also be written down compactly as a triple product:

$$L_6 = (\vec{\bar{j}}, v\vec{\bar{j}}, v^2\vec{\bar{j}} - 3\vec{j}) \tag{2.16}$$

involving the vectors $\vec{j}$, $\vec{\bar{j}}$ and the matrix v.

Finally, the eight variables are subject to an *algebraic constraint*

$$K_6 = 0 \tag{2.17}$$

whose detailed expression may be found in paper I (see equation (5.11) therein). K_6 has the property of being exactly invariant under the inversion of the principal axes; in addition, it is a *homogeneous* function of the sixth degree of the components of v.

3. The two-dimensional surface (Σ_2)

For the Liouville integrable Hamiltonian motion in five-dimensional space ($S_2 \times 0(3)$) with five commuting integrals of motion (m, α^2, I_6, L_6, J_z), the Liouville tori are five dimensional. However, the location in the space 0(3) being irrelevant as a consequence of 0(3) invariance, the corresponding sub-manifold (Σ_3) in the space (X_n, Y_n, T, P) has lower dimensionality. In the case of a fixed rotation axis, (Σ_3) may be simply identified with the sphere S_2, and is two dimensional. In general, however, when there is no fixed axis, the component j_z of angular momentum in the moving frame is no longer determined by the integral of motion J_Z, and (Σ_3) is three dimensional.

In the special case where the energy constant m takes its minimum value m_0 (the other constants α^2, I_6, L_6 being kept fixed), (Σ_3) may still be three dimensional as a complex manifold, but its restriction to real space becomes two dimensional, and is spanned by a one-parameter family of real trajectories, whose determination is the subject of the present work.

We will for simplicity restrict our consideration to a minimal energy case with a vanishing value of L_6. In such cases there exists one real trajectory which is precessionless, but all the others are precessing. This suggests that the cases where L_6 is non-zero might not be essentially different.

3.1. Characterization of the real surface (Σ_2)

When a three-dimensional algebraic hypersurface (Σ_3) has a real part reduced to two dimensions, that real surface (Σ_2) is algebraic as well. Thus in the minimal energy case there exists an algebraic combination of (X_n, Y_n, T, P) to be found, distinct from the constants of motion, and whose vanishing characterizes the set of real trajectories. It may be found in the following way.

We start with the observation that, in minimal energy cases where $L_6 = 0$, there exists on the Liouville torus a real solution without precession, which has been characterized in earlier works. For definiteness, we consider in what follows the particular set of constants:

$$\alpha^2 = 12 \qquad \varepsilon = 4 \qquad L_6 = 0$$

for which, as mentioned in paper II, the minimum value of the energy is

$$m_0 = 5.$$

The real solution without precession is found to satisfy the simple relations:

$$\begin{cases} Z_{12} \equiv A_{12} + 6(T - Y_0 + X_0) = 0 \\ Z_{22} \equiv A_{22} + 2\left(X_0^2 - 6Y_0 - 9\right) = 0 \end{cases} \tag{3.1}$$

which completely determine the trajectory, under the assumption of a fixed rotation axis. We considered the possibility that these equations might still be applicable to the precessing solutions as well, and might in fact constitute the equation of the surface (Σ_2) that we are looking for.

Assuming then that Z_{12} and Z_{22} vanish all over (Σ_2), their derivatives $Z'_{12}(u)$, $Z'_{22}(u)$ must of course also vanish; making use of the equations of motion, we obtain:

$$\begin{cases} \dfrac{-\mathrm{d}}{\mathrm{d}u}(\vec{j}\cdot\vec{\vec{j}}) = P(9 - X_0Y_0) - (X_1Y_2 + X_2Y_1) \\ \dfrac{\mathrm{d}}{\mathrm{d}u}(\vec{\vec{j}}^{\,2}) = 4P\big(3X_0 - Y_0^2\big) + 4X_1T + 4Y_1(Y_2 + TY_0) \end{cases} \tag{3.2}$$

and hence the derivatives of Z_{12}, Z_{22}:

$$\begin{cases} -\mathrm{d}Z_{12}/\mathrm{d}u = P(X_0Y_0 - 27) + X_1(Y_2 - 12) + Y_1(X_2 - 6) \\ \dfrac{1}{4}\dfrac{\mathrm{d}Z_{22}}{\mathrm{d}u} = P\big(3X_0 - Y_0^2\big) + X_1(T + 2X_0) + Y_1(Y_0T + Y_2 + 6). \end{cases} \tag{3.3}$$

Considering now the system:

$$\begin{cases} Z_{12} = Z_{22} = 0 \\ Z'_{12} = Z'_{22} = 0 \end{cases} \tag{3.4}$$

it turns out that these four equations are not independent (when m and α^2 are given); an independent sub-set consists, e.g., of

$$\begin{cases} Z_{12} = Z_{22} = 0 \\ Z'_{22} = 0. \end{cases} \tag{3.5}$$

Taking account of the m and α^2 integrals, the variables X_2 and Y_2 may be easily eliminated, and the system (3.5) may serve to determine (X_1, Y_1, P) as functions of (X_0, Y_0, T). The solution assumes the form of a bi-quadratic equation for X_1 (i.e. a second-degree one for X_1^2). Substituting it into the expressions of the integrals of motion, one finds that the following relations:

$$\begin{aligned} &\text{(a)}\quad K_6 = -L_6 \\ &\text{(b)}\quad \varepsilon - 4 = L_6/108 \end{aligned} \tag{3.6}$$

identically hold for arbitrary values of all three independent variables (X_0, Y_0, T).

This remarkable result shows that if we complete the system (3.5) by the equation

$$\varepsilon = 4 \tag{3.7}$$

we obtain a two-dimensional surface in (X_0, Y_0, T) space on which, in addition to m and α^2, the integrals ε, L_6 and K_6 all have their prescribed values; that is to say, the surface obtained is a real two-dimensional sub-manifold of (Σ_3), which may be identified with (Σ_2).

3.2. The tenth-degree surface (Σ_2) in coordinates (X_0, Y_0, T)

The equation of the surface in coordinates (X_0, Y_0, T) is found through the elimination of the remaining variables (X_1, Y_1, P), which can be done in the following way.

The system reads, in compact form

$$\begin{cases} \big(3X_1^2 - X_0Y_1^2\big) = A_1 \\ \big(3X_1Y_1 + Y_0Y_1^2\big) = -B_1 \\ \big(Y_0^2 - 3X_0\big)(P + Y_1) = X_1(T + 2X_0) - Y_1C_1 \end{cases} \tag{3.8}$$

where A_1, B_1, C_1 are polynomials in (X_0, Y_0, T):

$$\begin{cases} A_1 \equiv \frac{4}{3}TX_0\left(Y_0^2 - 3X_0\right) - X_0Y_0(X_0 - 23) - \left(2X_0^3 - 27X_0 + 135\right) \\ B_1 \equiv T\left(\frac{4}{3}Y_0^3 - 5X_0Y_0 + 27\right) - Y_0^2(X_0 - 23) - 2Y_0X_0^2 + 3X_0(X_0 - 5). \\ C_1 \equiv \frac{1}{3}Y_0T - Y_0^2 + 2X_0 + 5 \end{cases} \tag{3.9}$$

The system must be completed by the condition $\varepsilon = 4$, which is:

$$(P + Y_1)^2 = -S_3/3 \tag{3.10}$$

where

$$S_3 \equiv \frac{4}{9}T^3 - \frac{8}{3}T^2Y_0 + 4T\left(Y_0^2 - 2X_0 - 11\right) + 8\left(6Y_0 - X_0^2 + 15\right). \tag{3.11}$$

Then, introducing

$$\begin{aligned} &\text{(a)} \quad D_2 \equiv X_0(T + 2X_0)^2 + 2Y_0C_1(T + 2X_0) + 3C_1^2 \\ &\text{(b)} \quad E_2 \equiv (T + 2X_0)A_1 + 2B_1C_1 \\ &\text{(c)} \quad R_3 \equiv \left(Y_0^2 - 3X_0\right)^2 S_3 + (T + 2X_0)E_2 \end{aligned} \tag{3.12}$$

one obtains

$$Y_1^2 = -R_3/D_2 \tag{3.13}$$

and, in addition

$$X_1^2 = \frac{P_3}{3D_2} \tag{3.14}$$

$$X_1Y_1 = \frac{Q_3}{3D_2} \tag{3.15}$$

where

$$\begin{cases} P_3 \equiv (A_1D_2 - X_0R_3) \\ Q_3 \equiv (Y_0R_3 - B_1D_2). \end{cases} \tag{3.16}$$

Owing to a cancellation of the highest degree terms between A_1D_2 and X_0R_3, P_3 is of degree 7 only in (X_0, Y_0, T), as is R_3; and similarly Q_3 is of degree 7 only; the equation of the surface

$$\left(Q_3^2 + 3P_3R_3\right) = 0 \tag{3.17}$$

is accordingly of degree 14. There is, however, a simplification by a factor $\left(Y_0^2 - 3X_0\right)^2$, so that the final result

$$F(X_0, Y_0, T) \equiv \frac{\left(Q_3^2 + 3P_3R_3\right)}{\left(Y_0^2 - 3X_0\right)^2} = 0 \tag{3.18}$$

is of the tenth degree. The coefficients of F are listed in appendix A, using for convenience a slightly modified coordinate system (X_0, Y_0, S), where

$$S \equiv T - Y_0. \tag{3.19}$$

3.3. Asymptotic behaviour and conic point singularity

Introducing for conciseness the new variable S in place of T, the homogeneous terms of degree 10 in F amount to

$$F_{10} = X_0^2Y_0^2S^4(2X_0 + 3S)^2. \tag{3.20}$$

Both factors X_0 and $(2X_0 + 3S)$ in F_{10} are associated with a double line at infinity.

There is also a conic point, denoted (K_4), of order 4, located at

$$X_0 = -3 \qquad Y_0 = 1 \qquad S = 5.$$

It is a degenerate one, locally decomposable into four intersecting complex conjugate planes.

There is in addition a number of conic points of the second order.

3.4. Other double lines

There are two double lines in addition to the two at infinity: the first one is just the trajectory (L_0), the particular solution without precession.

There is another special line on the surface (although not a trajectory), whose existence is manifest from the form of equation (3.13): it is the line, denoted (L_2), defined by the equations:

$$D_2 = R_3 = 0 \tag{3.21}$$

and it is a double line too.

Both (L_0) and (L_2) pass through the conic point (K_4), which is why it is a degenerate conic point of the fourth order.

The singular solution (L_0) is a unicursal curve, and the slope $(S-5)/(X_0+3)$ of a straight line through the point (K_4) may serve as the parameter. It admits the following representation, in terms of an homographically related parameter r:

$$\begin{cases} X_0 = \frac{3(3r^3+3r+2)}{P_2(r)} \\ S = -\left(X_0 + \frac{4}{P_2}\right) \\ Y_0 = \left(\frac{X_0^2}{6} - \frac{3}{2} + \frac{4}{P_2^2}\right) \end{cases} \tag{3.22}$$

where

$$P_2(r) \equiv (3r^2+1). \tag{3.23}$$

The line (L_2) on the other hand is less easy to study. Its intersection with planes $X_0 =$ constant gives rise to a 13th degree equation, given in appendix B. We note that the section by the plane $X_0 = 6$ decomposes into a set of four points plus another of nine points.

3.5. Straight lines

We have seen that the section of the surface by the plane at infinity fully decomposes into straight lines, of which two are double lines. Curiously, there exists yet another straight line on the surface, defined by the equations

$$\begin{cases} X_0 = 7 \\ T = -2/3. \end{cases} \tag{3.24}$$

There are six double points aligned on it: two from the singular solution (L_0), and four from (L_2).

Taking Y_0 as a free parameter, the remaining variables X_1, Y_1, P take the following values:

$$\begin{cases} X_1 = \pm 2\sqrt{2}/3 \\ Y_1 = X_1(Y_0 - 9) \\ P = 2X_1/9. \end{cases}$$

4. A parametric representation of the surface

A natural way of finding a parametric representation of a surface is to look for plane (or straight line) sections of a type sufficiently simple that they can be parametrized.

Let us first consider the linear sections through the point (K_4): there are in general six points of intersection with the surface, which is intractable. If the line considered also intersects the second double line at infinity, i.e. if it lies in the plane:

$$(S + 2X_0/3) = 3 \tag{4.1}$$

there are only four other intersections; moreover, the corresponding quartic equation decomposes into a pair of trinomials, so that particular plane section does admit a simple parametrization.

In view of the existence of the first double line at infinity (in the direction of fixed X_0) and of the special physical meaning of X_0 as one of the coordinates on the sphere S_2, it seems natural to try next the plane sections $X_0 =$ constant, which are curves of degree 8. In the particular case, where $X_0 = 6$, the section turns out to have the form

$$F \equiv A_4^2 - (2Y_0 + 1)A_3^2 = 0 \tag{4.2}$$

where A_3 and A_4 are, respectively, cubic and quartic polynomials in Y_0 and S; it is thus decomposable, through the transformation:

$$Y_0 = (y^2 - 1/2) \tag{4.3}$$

into a pair of curves of the sixth degree.

Only four of the 13 points of (L_2) remain double points under that transformation, which explains the decomposability of the equation of (L_2) when $X_0 = 6$, mentioned earlier.

In the same way, only one point of (L_0) remains double, so there are five double points in all; and there is also a triple point at infinity. If we then cut by a family of unicursal cubics[1] having their double point coinciding with the triple point and passing through the five double points, there only remain two movable intersections, and a parametrization is found which merely involves a square root.

A similar result may be obtained for arbitrary sections $X_0 =$ constant, in spite of the lack of decomposability into lower degree curves when $X_0 \neq 6$. In general, there are 18 double points in all: 13 on (L_2), three on (L_0) plus two double points at infinity: the double point (M_1) in direction Y_0 is fixed, and another double point (M_2) in direction S is fixed; and in the latter case, the two branches have a common tangent, which is the line at infinity. Cutting by a family of quintic (fifth degree) curves passing through all the double points, and admitting the common tangent at (M_2), there are only two movable points of intersection[2], and a parametrization of the section, involving a square root, is obtained. It has the following form:

$$\begin{cases} S = \frac{[G_1 + 4(w-3)\delta_2\delta_4]}{(w+3)^2 A_2} \\ Y_0 = \frac{[F_2 + 12F_1\delta_2\delta_4]}{6(w+3)^2 A_2^2} \end{cases} \tag{4.4}$$

where

$$\begin{cases} \delta_2 \equiv \sqrt{-2\Delta_2} \\ \delta_4 \equiv \sqrt{-\Delta_4/3} \end{cases} \tag{4.5}$$

[1] A unicursal curve admits a rational parametrization. A unicursal cubic must have one double point.

[2] The sections of the surface by planes $X_0 =$ constant are of degree 8, since these planes pass through a double straight-line at infinity; thus there are $8 \times 5 = 40$ intersections with a quintic. These quintics are non-decomposable in general (the 13 points on (L_2) are not aligned), and they present 38 fixed intersections (among which four intersections at M_2), whence the remaining two movable intersections. An account of this method may be found in Goursat (1949, vol 2, ch 15).

and Δ_2, G_1, F_1, F_2 are polynomials in X_0 and w, while Δ_4 and A_2 are functions of w only:

$$A_2(w) \equiv w^2 - 42w + 201 \tag{4.6}$$

$$\Delta_4(w) \equiv w^4 + 6w^3 + 18w^2 + 54w - 207 \tag{4.7}$$

$$\Delta_2(X_0, w) \equiv X_0^2 u_2(w) + 12X_0(w-1)(w+3) + 30(w+3)^2 \tag{4.8}$$

with

$$\begin{aligned} u_2(w) &\equiv w^2 + 6w - 39 \\ G_1(X_0, w) &\equiv X_0 d_2(w) + 4d_4(w) \end{aligned} \tag{4.9}$$

with

$$\begin{cases} d_2(w) \equiv 3w^4 + 20w^3 - 78w^2 - 1068w + 99 \\ d_4(w) \equiv 5w^4 - 246w^2 + 504w - 4743. \end{cases}$$

The expressions of F_1 and F_2, which are, respectively, linear and quadratic in X_0, are given in appendix D.

The sections w = constant of the surface are thus conic sections in projection on the plane (X_0, S):

$$\Phi(S, X_0, w) \equiv \left(c_1 S^2 + c_2 X_0 S + c_3 X_0^2\right) + (c_4 S + c_5 X_0) + c_6 = 0 \tag{4.10}$$

where the c_n (see appendix D) are functions of w only.

Using the above results, the variables X_1, Y_1 and P may also be determined in terms of X_0 and w; we give below the expression obtained for X_1:

$$-6(w+3)A_2 X_1/\sqrt{3} = 4\delta_4 X_0[(w-9)X_0 + 6(w-11)] - \delta_2(X_0 v_2 + 3A_2) \tag{4.11}$$

where

$$v_2(w) \equiv w^2 + 10w - 27.$$

The parameter w is only determined by the present method up to an arbitrary homographic (Möbius) transformation, which can be chosen independently for each value of X_0. However, it turned out to be possible to choose it in such a way that the discriminant Δ_4 did not depend on X_0. Moreover, the product $(w-3)^2\Delta_4(w)$ precisely coincides with the sixth degree polynomial $A_6(w)$ introduced in paper II (see equation (2.35) therein) when the present values of the integrals of motion are substituted. This is a remarkable result, in view of the fact that the analysis of paper II merely concerned solutions in rotation around a fixed axis, whereas the parametric representation given here concerns solutions which are all precessing, with the only exception of the singular solution (L_0).

5. Separation of variables and linearization

5.1. Separability of the variable w

The parametric representation enables us to calculate the derivatives of X_0, Y_0 and S as functions of (X_0, w) using the general formulae:

$$\begin{cases} X_0'(u) = 2X_1 \\ Y_0'(u) = -2Y_1 \\ S'(u) = 3P + Y_1. \end{cases} \tag{5.1}$$

The derivative of w may then be obtained through differentiation of equation (4.10):

$$\Phi_x X_0'(u) + \Phi_s S'(u) + \Phi_w w'(u) = 0 \tag{5.2}$$

and, as it turns out, it is given by the very simple formula:

$$w'(u) = \delta_4/\sqrt{3} \tag{5.3}$$

which shows that the evolution of w can be determined independently of that of the remaining variables; moreover, the function $w(u)$ is elliptic.

The problem is now reduced to that of solving a first-order ordinary differential equation (o.d.e.) for the function $X_0(w)$; using: $\frac{dX_0}{dw} = \frac{2X_1}{w'(u)}$, where X_1 is given by equation (4.11), the equation is the following:

$$(w+3)A_2\frac{dX_0}{dw} + 4X_0[(w-9)X_0 + 6(w-11)] = \frac{\delta_2}{\delta_4}(v_2X_0 + 3A_2). \tag{5.4}$$

5.2. *Reduction to the Riccati form*

Assuming that the Painlevé property still holds in the presence of precession, we expect that $X_0(u)$ has only pole singularities and hence that $X_0(w)$ has no movable singularities other than poles. If the o.d.e. for $X_0(w)$ did not include an irrational dependence on X_0 (through the factor δ_2), it would have to be of Riccati type (Ince 1956); this suggests that a rationalizing transformation eliminating the square root δ_2 would give rise to an equation of Riccati form for the new unknown function.

Introducing the rescaling

$$\frac{X_0}{(w+3)} \equiv \beta \tag{5.5}$$

the expression of Δ_2 simplifies to

$$\begin{aligned}\frac{\Delta_2}{(w+3)^2} &\equiv \beta^2 u_2(w) + 12\beta(w-1) + 30 \\ &\equiv u_2(\beta-\beta_0)(\beta-\beta_1)\end{aligned} \tag{5.6}$$

where β_0, β_1 are the roots of the trinomial, and are expressed by

$$\beta_0 = \frac{\left[6(1-w) + \sqrt{6A_2}\right]}{u_2} \tag{5.7}$$

(together with a similar expression for β_1). A rationalizing transformation is obtained through the introduction of the variable λ:

$$\lambda = \frac{\delta_2}{(w+3)(\beta-\beta_0)}. \tag{5.8}$$

Both δ_2 and X_0 then have rational expressions in terms of λ:

$$\begin{cases} X_0 = (w+3)\beta \qquad \text{with} \\ \beta - \beta_0(w) = -4\sqrt{6A_2}/(\lambda^2 + 2u_2) \end{cases}$$

for any given value of the independent variable w.

The resulting o.d.e. for $\lambda(w)$ has the Riccati form, as expected, and reads

$$2u_2A_2\frac{d\lambda}{dw} = \left[2\lambda A_2(w+3) + \frac{3v_4(\lambda^2+2u_2)}{(w+3)\delta_4}\right] + \sqrt{6A_2}\left[8\lambda(w-9) - \frac{v_2(\lambda^2-2u_2)}{\delta_4}\right] \tag{5.10}$$

where

$$\begin{aligned} v_4(w) &\equiv 2v_2(w-1)(w+3) - A_2u_2 \\ &\equiv w^4 + 60w^3 + 70w^2 - 3012w + 8001. \end{aligned}$$

5.3. Linearization

The Riccati equation can be linearized, since one particular solution is known: the singular solution (L_0). The parameter r which occurs in equation (3.22) describing the solution (L_0) is an elliptic function of u, defined by

$$\begin{cases} r'^2(u) \equiv -\frac{2}{9}P_4(r) \\ P_4(r) \equiv 9r^4 - 18r^3 + 9r^2 - 12r - 2. \end{cases} \tag{5.11}$$

As the roots of $P_4(r)$ are homographically related to those of $\Delta_4(w)$, there must exist an algebraic relation between r and w, of second degree both in r and in w:

$$3r^2M_2(w) + 24r(w-3) + 4(w^2-3) = 0 \tag{5.12}$$

where

$$M_2(w) \equiv w^2 + 6w + 33.$$

Solving for r or for w, one finds:

$$r = -2[2(w-3) + \delta_4]/M_2(w) \tag{5.13}$$

$$w = \frac{-\left[3r(3r+4) + 2\sqrt{-6P_4(r)}\right]}{(3r^2+4)}. \tag{5.14}$$

Equations (5.12), (5.13) and (5.14) may be viewed as the Miura transformation, applied to elliptic functions.

Substituting the expression (5.13) of $r(w)$, the corresponding expressions of X_0, of δ_2, δ_4 and of λ in terms of w may be obtained; λ is found to satisfy the second degree equation:

$$\lambda^2\left[3H_5 - 2(w+3)H_3\sqrt{6A_2}\right] - 36\lambda\delta_4u_2H_2 - 2u_2\left[3H_5 + 2(w+3)H_3\sqrt{6A_2}\right] = 0 \tag{5.15}$$

where

$$\begin{cases} H_2 \equiv (w^2 - 10w + 73) \\ H_3 \equiv (w^3 - 3w - 126) \end{cases}$$

and H_5 is a fifth degree polynomial in w.

Let us note that the coefficient of λ^2 in equation (5.15) admits the irrational decomposition:

$$10\left[3H_5 - 2(w+3)H_3\sqrt{6A_2}\right] \equiv \left[6(w-1) - \sqrt{6A_2}\right]\left[2K_4 - K_3\sqrt{6A_2}\right] \tag{5.16}$$

where

$$\begin{cases} K_3(w) \equiv w^3 + 27w^2 - 9w + 429 \\ K_4(w) \equiv 7w^4 - 48w^3 + 78w^2 - 2664w - 2493. \end{cases}$$

Solving for λ, one obtains the two roots λ_A and λ_B:

$$\begin{cases} \lambda_A = \frac{u_2[18H_2\delta_4 - (w+3)^3\sqrt{6A_2}]}{[3H_5 - 2(w+3)H_3\sqrt{6A_2}]} \\ \lambda_B = \frac{u_2[18H_2\delta_4 + (w+3)^3\sqrt{6A_2}]}{[3H_5 - 2(w+3)H_3\sqrt{6A_2}]}. \end{cases} \tag{5.17}$$

Both roots λ_A and λ_B constitute particular solutions of the Riccati equation (5.10), which is thus not only linearizable, but also integrable by quadrature; and its general solution must have the form:

$$\ln\left(\frac{\lambda - \lambda_A}{\lambda - \lambda_B}\right) = \int f(w)\,\mathrm{d}w. \tag{5.18}$$

We obtain the following expression for the integrand $f(w)$:

$$f(w) \equiv \frac{6(w+3)^2}{\delta_4}\left[(w+3)N_5 - \frac{N_7}{\sqrt{6A_2}}\right]\bigg/ D_8 \tag{5.19}$$

where N_5, N_7 and D_8 are polynomials in w of degrees 5, 7 and 8, respectively. $D_8(w)$ admits the irrational decomposition

$$10D_8 \equiv \left(2K_4 - K_3\sqrt{6A_2}\right)\left(2K_4 + K_3\sqrt{6A_2}\right)$$

and the factor at the numerator may be similarly decomposed as

$$10\left[N_7 - (w+3)N_5\sqrt{6A_2}\right] \equiv \left(2K_4 + K_3\sqrt{6A_2}\right)\left(4N_3 - A_2\sqrt{6A_2}\right)$$

where

$$N_3(w) \equiv w^3 + 97w^2 - 357w + 99$$

so that the expression of the integrand may be simplified to

$$f(w) \equiv \frac{-6(w+3)^2}{\delta_4\sqrt{6A_2}} \frac{\left(4N_3 - A_2\sqrt{6A_2}\right)}{\left(2K_4 - K_3\sqrt{6A_2}\right)}. \tag{5.20}$$

This is not an integral of elliptic type, owing to the presence of the radical $\sqrt{6A_2}$, in addition to δ_4. However, the extra singularities occurring as A_2 vanishes are introduced by the rationalizing transformation (5.7), (5.8) and (5.9), which explicitly involves $\sqrt{6A_2}$; these singularities can be made to disappear through an appropriate (w dependent) Möbius transformation on λ. One way to see it is to consider the linearized equations of motion in the neighbourhood of the solution (L_0).

5.4. Reduction to an elliptic integral

The singular solution (L_0) may be written in the form:

$$\Psi(X_0, w) \equiv H_5X_0 + 2(w+3)K_4 + 3H_2\delta_2\delta_4 = 0. \tag{5.21}$$

The o.d.e. (5.4) for $X_0(w)$ gives rise to the following generally valid expression for the derivative of Ψ:

$$-(w+3)A_2\frac{\mathrm{d}\ln\Psi}{\mathrm{d}w} = [4X_0(w-9) - b_2(w)] - \frac{\delta_2}{\delta_4}v_2(w) \tag{5.22}$$

where

$$b_2(w) \equiv 5w^2 - 186w + 741.$$

Let us now substitute on the right-hand side of equation (5.22) the values that X_0 and δ_2 take at the singular solution. The expression of X_0 may be deduced from (5.21) and reads:

$$X_0 = -\frac{2}{M_6}[H_6 + 3(w+3)^2H_2\delta_4] \tag{5.23}$$

where

$$M_6(w) \equiv M_2M_4.$$

$M_2(w)$ is given by equation (5.12),

$$M_4(w) \equiv 3w^4 - 12w^3 - 14w^2 - 204w + 1251 \tag{5.24}$$

and

$$H_6(w) \equiv 6(w-3)M_4 + H_4M_2 \tag{5.25}$$

with

$$H_4(w) \equiv 3w^4 + 32w^3 - 114w^2 + 120w - 2601.$$

We note the identity:

$$3M_6 \equiv 8H_3^2 + (w+3)^4 u_2. \tag{5.26}$$

The corresponding value of $\delta_2(X_0, w)$ is then:

$$\delta_2 = \frac{2}{M_6}[(w+3)^2 H_5 + 12 H_2 H_3 \delta_4]. \tag{5.27}$$

Upon substitution of (5.23), (5.27) into equation (5.22), and introducing for convenience the rescaling:

$$\Psi \equiv (w+3)^6 A_2 \hat{\Psi} \tag{5.28}$$

one obtains the following expression, valid in the neighbourhood of $\hat{\Psi} = 0$:

$$\frac{\mathrm{d}\ln\hat{\Psi}}{\mathrm{d}w} = \frac{1}{2}\frac{\mathrm{d}\ln M_6}{\mathrm{d}w} - \frac{2(w+3)B_5}{M_6\delta_4} \tag{5.29}$$

where $B_5(w)$ is the fifth degree polynomial defined by:

$$A_2 B_5 \equiv 4(w-9)H_2\Delta_4 - v_2 H_5 \tag{5.30}$$

i.e.

$$B_5(w) \equiv w^5 - 15w^4 - 94w^3 - 246w^2 - 1731w + 5157.$$

The integration of $\hat{\Psi}$ is thus reduced to one of elliptic type; and the term $\frac{1}{2}\frac{\mathrm{d}\ln M_6}{\mathrm{d}w}$ on the right-hand side of equation (5.29) merely serves to cancel out the branch-point singularities occurring at $M_6 = 0$ in the elliptic integral. The cancellation occurs as a consequence of the exact divisibility by M_6 of the polynomial:

$$\Delta_4 M_6'^2(w) + 48(w+3)^2 B_5^2.$$

6. Conclusion

In the framework of Ovsiannikov and Dyson's model of rotating and expanding gas clouds, we have succeeded in obtaining the first complete determination of a family of solutions with precession (see appendix E). The mathematical expression of the result in its final form surprisingly resembles that for the precessionless cases; which is itself similar to the form of Kowalevski's (1889, 1890) results, as noted in paper II (p 9208 therein). In particular, the *same* elliptic function $w(u)$ is found to govern the flows with or without precession, at least under the restricting conditions imposed here; and in both cases the remaining dependent variable (X_0 say) is obtainable through the solution of the Riccati equation, whose integration is reducible to one quadrature of elliptic type.

These results suggest that there may exist a deep unity underlying the mathematical descriptions of the ellipsoidal gas flows with and without precession. Let us also mention the identification obtained here of the last integral of motion L_6, with a triple product of a simple form (equation (2.16)).

Appendix A. The equation of the surface (Σ_2)

Writing the equation of the tenth degree surface (Σ_2) in the form:

$$F(X_0, Y_0, S) \equiv \sum_{ijk} c_{ijk} X_0^i Y_0^j S^k = 0.$$

We list below (table 1) the non-vanishing coefficients c_{ijk} which all are integers.

Table 1. We list here the coefficients $c(i,\ j, k)$ of the polynomial function $F(X_0, Y_0, S)$ as defined in the text.

i	j	k	$c(i, j, k)$
Degree 10			
2	2	6	27
3	2	5	36
4	2	4	12
Degree 9			
1	3	5	378
2	3	4	252
3	1	5	162
3	3	3	54
4	0	5	72
4	1	4	324
4	3	2	324
5	0	4	96
5	1	3	216
5	3	1	384
6	0	3	32
6	1	2	48
6	3	0	128
Degree 8			
0	4	4	1 323
1	1	6	−162
1	2	5	486
1	4	3	−2 808
2	1	5	−54
2	2	4	2 592
2	4	2	−4 950
3	1	4	1152
3	2	3	3 780
3	4	1	−2 592
4	0	4	243
4	1	3	528
4	2	2	1 674
4	4	0	−165
5	1	2	−1 440
5	2	1	432
6	0	2	−648
6	1	1	−1 728
6	2	0	216
7	0	1	−576
7	1	0	−576
8	0	0	−144
Degree 7			
0	2	5	−486
0	3	4	1 458
0	5	2	−17 064
1	2	4	8 046
1	3	3	−3 726
1	5	1	−17 280
2	0	5	−486
2	3	2	−8 370
2	5	0	−10 008
3	0	4	2 754
3	1	3	17 010
3	2	2	23 382
3	3	1	−18 738
4	0	3	3 996
4	1	2	17 496
4	2	1	12 852
4	3	0	−7 992
5	0	2	3 024
5	1	1	5 832
5	2	0	648
6	0	1	−144
6	1	0	288
7	0	0	−864
Degree 6			
0	0	6	243
0	1	5	−1 458
0	2	4	2 187
0	3	3	37 044
0	4	2	−31 752
0	6	0	21 168
1	0	5	1 134
1	1	4	12 150
1	2	3	26 244
1	3	2	44 604
1	4	1	21 384
2	0	4	9 207
2	1	3	95 904
2	2	2	90 639
2	3	1	41 796
2	4	0	−13 392
3	0	3	29 232
3	1	2	118 908
3	2	1	61 236
3	3	0	54 828
4	0	2	40 500
4	1	1	107 352
4	2	0	10 368
5	0	1	25 056
5	1	0	35 856
6	0	0	9 072
Degree 5			
0	1	4	1 620
0	2	3	85 050
0	3	2	−7 290
0	4	1	− 231 120
0	5	0	181 440
1	0	4	−8 748

Table 1. (Continued.)

i	j	k	$c\,(i, j, k)$	i	j	k	$c\,(i, j, k)$
2	1	4	4 374	1	1	3	69 606
2	2	3	22 950	1	2	2	223 074
1	3	1	217 728	Degree 3			
1	4	0	−276 912	0	0	3	52 650
2	0	3	48 114	0	1	2	643 950
2	1	2	157 698	0	2	1	−72 900
2	2	1	−59 562	0	3	0	−156 600
2	3	0	53 946	1	0	2	484 110
3	0	2	67 554	1	1	1	443 070
3	1	1	129 816	1	2	0	− 2182 140
3	2	0	65 934	2	0	1	318 600
4	0	1	75 816	2	1	0	823 500
4	1	0	−23 976	3	0	0	245 160
5	0	0	26 568				
				Degree 2			
Degree 4				0	0	2	492 075
0	0	4	−21 870	0	1	1	25 650
0	1	3	43 740	0	2	0	− 3233 925
0	2	2	172 260	1	0	1	255 150
0	3	1	−508 680	1	1	0	− 2612 250
0	4	0	592 920	2	0	0	346 275
1	0	3	38 880				
1	1	2	503 010	Degree 1			
1	2	1	376 650	0	0	1	−182 250
1	3	0	− 1163 160	0	1	0	− 4556 250
2	0	2	114 372	1	0	0	− 1437 750
2	1	1	28 674				
2	2	0	508 302	Degree 0			
3	0	1	135 972	0	0	0	− 2008 125
3	1	0	145 800				
4	0	0	6 912				

Appendix B. The double line (L_2)

In section 3, we have noted the existence of a special line (L_2) on the surface (Σ_2), defined by the pair of equations:

$$D_2(X_0, Y_0, S) = R_3(X_0, Y_0, S) = 0. \tag{B.1}$$

It is the intersection of a quartic surface ($D_2 = 0$) with one of the seventh degree ($R_3 = 0$), and is a double line of the surface. Through the introduction of a new coordinate t:

$$t = Y_0 + \sqrt{Y_0^2 - 3X_0} \tag{B.2}$$

it is representable by an equation of degree 14 in (X_0, t):

$$R_{13}(X_0, t) \equiv c_{13}t^{13} + \cdots + c_0 = 0 \tag{B.3}$$

where

$$c_{13} \equiv 3(2X_0 + 7)$$
$$c_{12} \equiv 4\left(X_0^2 - 15\right)$$

$$c_{11} \equiv -3\left(35X_0^2 + 187X_0 + 320\right)$$
$$c_{10} \equiv -18\left(8X_0^3 + 28X_0^2 + 6X_0 - 135\right)$$
$$c_9 \equiv -3\left(16X_0^4 - 233X_0^3 - 1854X_0^2 - 4080X_0 - 2000\right)$$
$$c_8 \equiv 18\left(108X_0^4 + 496X_0^3 + 91X_0^2 - 2700X_0 - 5400\right)$$
$$c_7 \equiv 54X_0\left(24X_0^4 + X_0^3 - 567X_0^2 - 1740X_0 - 3400\right)$$
$$c_6 \equiv 12\left(16X_0^6 - 972X_0^5 - 3780X_0^4 + 3420X_0^3 + 25\,515X_0^2 + 48\,600X_0 + 81\,000\right)$$
$$c_5 \equiv -27X_0^2\left(432X_0^4 + 652X_0^3 - 5427X_0^2 - 20\,520X_0 + 7200\right)$$
$$c_4 \equiv -108X_0^2\left(32X_0^5 - 243X_0^4 - 1080X_0^3 - 2790X_0^2 + 7290X_0 + 24\,300\right)$$
$$c_3 \equiv 243X_0^3\left(144X_0^4 + 17X_0^3 - 1863X_0^2 - 1080X_0 - 7200\right)$$
$$c_2 \equiv 486X_0^4\left(32X_0^4 - 1188X_0^2 + 306X_0 + 3645\right)$$
$$c_1 \equiv -3^7X_0^5\left(103X_0^2 - 1080\right)$$
$$c_0 \equiv 3^{10} \times 14X_0^6.$$

Given a point (X_0, t) on that curve, the coordinates Y_0 and T are expressed by

$$\begin{cases} Y_0 = \dfrac{(t^2 + 3X_0)}{2t} \\ T = \dfrac{3}{2t}\dfrac{\left[t^4 - 2t^2(X_0 + 10) - 8tX_0^2 + 9X_0^2\right]}{(t^2 + 9X_0)}. \end{cases} \tag{B.4}$$

Appendix C. The family of quintic curves passing through all the double points

In section 4, we have seen that the plane sections $X_0 =$ constant of the surface (Σ_2) admit a parametric representation, which may be found through the consideration of the family of quintic curves that pass through all the 18 double points: 13 on the line (L_2), three on (L_0) and the remaining two at infinity. The form of these quintics is *a priori* arbitrary, except that the terms of the highest (fifth) degree must contain a factor Y_0S^2.

We give here, as an illustration of the method, the form of the result in the case of the section $X_0 = 3$. The family is generated by linearly combining the equations of any two members, Q_0 and Q_1 say; and when $X_0 = 3$ one may choose for Q_0 and Q_1 the following:

$$\begin{aligned} Q_0 \equiv Y_0S^2\left(Y_0^2 + 3S^2\right) &+ \left(-28Y_0^4 + 60Y_0^3S + 65Y_0^2S^2 + 66Y_0S^3 + 9S^4\right) \\ &+ \left(-17Y_0^3 + 56Y_0^2S + 279Y_0S^2 + 138S^3\right) + \left(219Y_0^2 + 178Y_0S + 555S^2\right) \\ &+ (-586Y_0 + 702S) - 164 = 0 \\ Q_1 \equiv Y_0S^2\left(3Y_0^2 + 5Y_0S\right) &+ \left(28Y_0^4 - 24Y_0^3S - 18Y_0^2S^2 - 24Y_0S^3 - 6S^4\right) \\ &+ \left(13Y_0^3 - 5Y_0^2S - 67Y_0S^2 - 69S^3\right) + \left(-446Y_0^2 - 136Y_0S - 258S^2\right) \\ &+ (-405Y_0 - 467S) - 428 = 0. \end{aligned}$$

Cutting the section $X_0 = 3$ of the surface (Σ_2) by a quintic which is an arbitrary linear combination of Q_0 and Q_1, one obtains, after simplification, *a second degree equation* for the two movable points of intersection.

Appendix D. The parametric representation of the surface

A parametric representation has been obtained, through the method described in section 4 and appendix C, which gives explicit expressions (equation (4.4)) for Y_0 and S in terms of X_0 and of another free parameter w.

In the form (4.10), the expression of $S(X_0, w)$ involves six coefficients $c_1, \ldots, c_6$ which are all sixth degree polynomials in w:

$$\begin{cases} c_1(w) \equiv 3(w+3)^4 A_2(w) \\ c_2(w) \equiv -6(w+3)^2 d_2(w) \\ c_3(w) \equiv -(w+3)^3 d_3(w) \\ c_4(w) \equiv -24(w+3)^2 d_4(w) \\ c_5(w) \equiv -24(w^6 - 26w^5 - 357w^4 - 1308w^3 - 4185w^2 - 25\,578w + 2781) \\ c_6(w) \equiv 48(5w^6 + 90w^5 + 315w^4 + 1260w^3 + 11\,691w^2 - 486w + 113\,589) \end{cases} \tag{D.1}$$

where $A_2(w)$, $d_2(w)$, $d_4(w)$ are given by equations (4.6) and (4.9), and

$$d_3(w) \equiv (5w^3 - 3w^2 - 681w + 423).$$

The expression (4.4) of Y_0 (X_0, w) involves two polynomials F_1 and F_2, respectively linear and quadratic in X_0:

$$F_1(X_0, w) \equiv X_0(w-9)v_2(w) + v_3(w) \tag{D.2}$$

where $v_2(w)$ is given by equation (4.11), and

$$v_3(w) \equiv 7w^3 - 69w^2 + 261w - 2439.$$

The expression of F_2 may be written:

$$F_2(X_0, w) \equiv X_0^2 f_2(w) + X_0 f_1(w) + f_0(w) \tag{D.3}$$

where

$$\begin{cases} f_2(w) \equiv (11w^6 - 18w^5 + 309w^4 - 1404w^3 - 10\,683w^2 + 141\,102w - 219\,429) \\ f_1(w) \equiv 12(11w^6 - 32w^5 - 333w^4 + 2040w^3 + 22\,401w^2 - 38\,232w + 137\,025) \\ f_0(w) \equiv 3(125w^6 - 390w^5 - 6549w^4 + 44\,748w^3 + 236\,547w^2 + 618\,138w + 1278\,261). \end{cases}$$

Appendix E. The (non-vanishing) rate of precession

The rate of precession of the angular momentum vector $\vec{j}$ in the moving frame is given by

$$\frac{\mathrm{d}\vec{j}}{\mathrm{d}t} = \vec{j} \wedge \vec{\omega} \tag{E.1}$$

(expressing conservation of the cloud's angular momentum), where the angular velocity vector $\vec{\omega} = (\omega_1, \omega_2, \omega_3)$ is related to $\vec{j}$ by

$$\omega_i = \beta_i j_i \qquad (i = 1, 2, 3) \tag{E.2}$$

where

$$\beta_1 = \frac{(\Delta_2 + \Delta_3)}{(\Delta_2 - \Delta_3)^2} \text{ (and circular permutation of the indices)}$$

and

$$\Delta_i \equiv D_i^2 \qquad (i = 1, 2, 3).$$

Thus the rate of precession cannot vanish unless $\vec{j}$ and $\vec{\omega}$ are parallel, which in turn requires that either $\beta_1 = \beta_2 = \beta_3$, or that the vector $\vec{j}$ has only one non-zero component.

In the first case, the condition $\beta_2 = \beta_3$ sub-divides into two sub-cases: either $\Delta_2 = \Delta_3$, or $\Delta_1 = \frac{(X_0 Y_0 + 3)}{4Y_0}$; thus the constraint $\beta_1 = \beta_2 = \beta_3$ is clearly too special to be satisfied all over the sub-manifold (Σ_2) of phase-space considered here.

The remaining possibility that j is parallel to a principal axis and the matrix v block-diagonal entails that $\vec{j}$ and $\vec{\tilde{j}}$ are parallel, and hence: $(\vec{j} \cdot \vec{\tilde{j}})^2 = (\vec{j}^2)(\vec{\tilde{j}})^2$, i.e. (see section 3.1):

$$A_{12}^2 = \alpha^2 A_{22}. \tag{E.3}$$

Substituting the expressions (3.1) of A_{12}, A_{22}, the condition becomes:

$$3(X_0 + S)^2 + 2(X_0^2 - 6Y_0 - 9) = 0 \tag{E.4}$$

and is indeed valid on the singular solution (L_0). However, it is clearly not valid at arbitrary points on (Σ_2), where the only applicable constraint on the three coordinates is (appendix A):

$$F(X_0, Y_0, S) = 0.$$

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